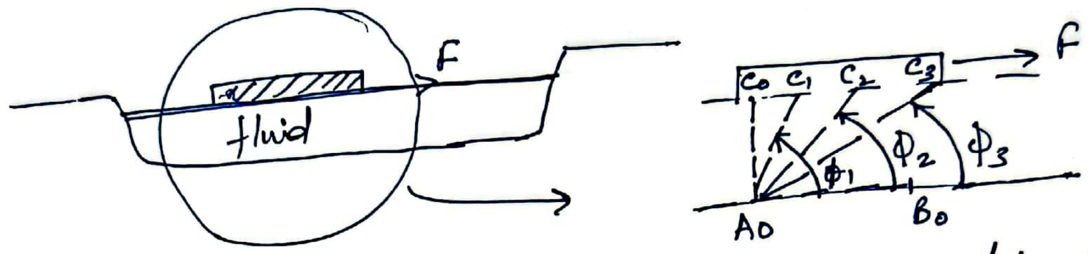
**Definition of a fluid:** A fluid is a substance that deforms continuously in time in response to the smallest shear stress.

**Remarks**

i) Deformation is continuous in time even if the magnitude of stress is held constant. Therefore it is meaningful to define strain rates rather than strains (as in case of solids).

ii) The definition is ~~sitient~~ silent on the behaviour of fluid in response to normal stresses.

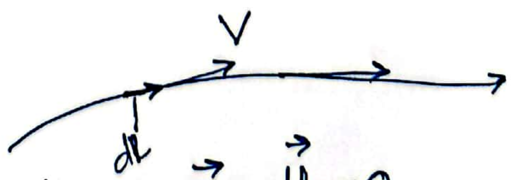



Continuous shear deformation under the action of constant shear.

## Kinematics

### Flow Visualization

① **Stream lines:** → A curve drawn through different points of the flow domain such that the velocity of fluid particle at all the points on the curve at a given instant of time is tangent to the curve.




Mathematically $\vec{V} \times \vec{dl} = 0$

Remarks:-

i) Stream lines given an instant snapshot of the flow field.

ii) Stream line must start/end at flow boundaries or form closed curves.

iii) No two stream lines can intersect at points of finite non-zero velocity.

In Cartesian: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

or $\boxed{\frac{dy}{dx} = \frac{v}{u}, \quad \frac{dz}{dx} = \frac{w}{u}}$

system of o.d.e's

So individual stream lines can be found by integrating the ode's subject to initial condition ~~(x,y,z)~~ $(x_0, y_0, z_0)$.

2) Path lines:- A path line is a trajectory of an individual fluid particle over an interval of time.

$$\frac{d\vec{r}}{dt} = \vec{V}(\vec{r}, t) \Rightarrow \text{solution } \vec{r}(t)$$

The fluid particle is identified by its location $\vec{r}_0$ at some time instant '$t_0$'.

$$\text{i.e. } \vec{r}(t_0) = \vec{r}_0$$

Remarks

i) Path lines given flow history over a period of time

ii) Path lines can intersect.

Scanned with CamScanner

③ Streak lines :- A streakline is the curve obtained by joining the instantaneous locations of a set of fluid particles at a given instant of time ('to') that have earlier ($t < t_0$) passed through a given point $\vec{r}_0$.

$$\frac{d\vec{r}}{dt} = \vec{V}(\vec{r}, t)$$

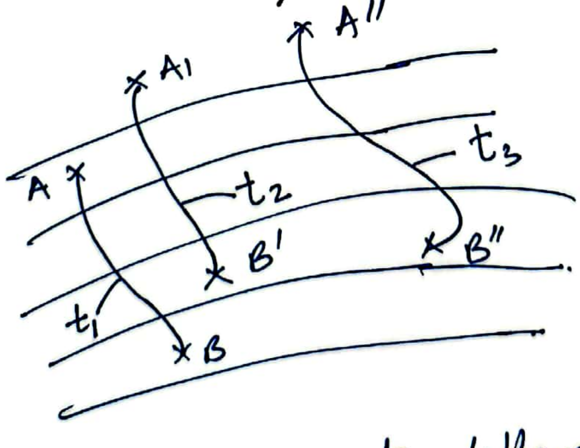
The solution of the above equation at $t = t_0$ subject to the parametrized ~~individual~~ initial condition.

$$\vec{r}(t < t_0) = \vec{r}_0$$

$$\vec{r}(t_0) - \vec{r}_0 = \int_{t \leftarrow \text{Variable}}^{t_0} \vec{V}(r, t)\,dt$$

④ Time line / Material curve

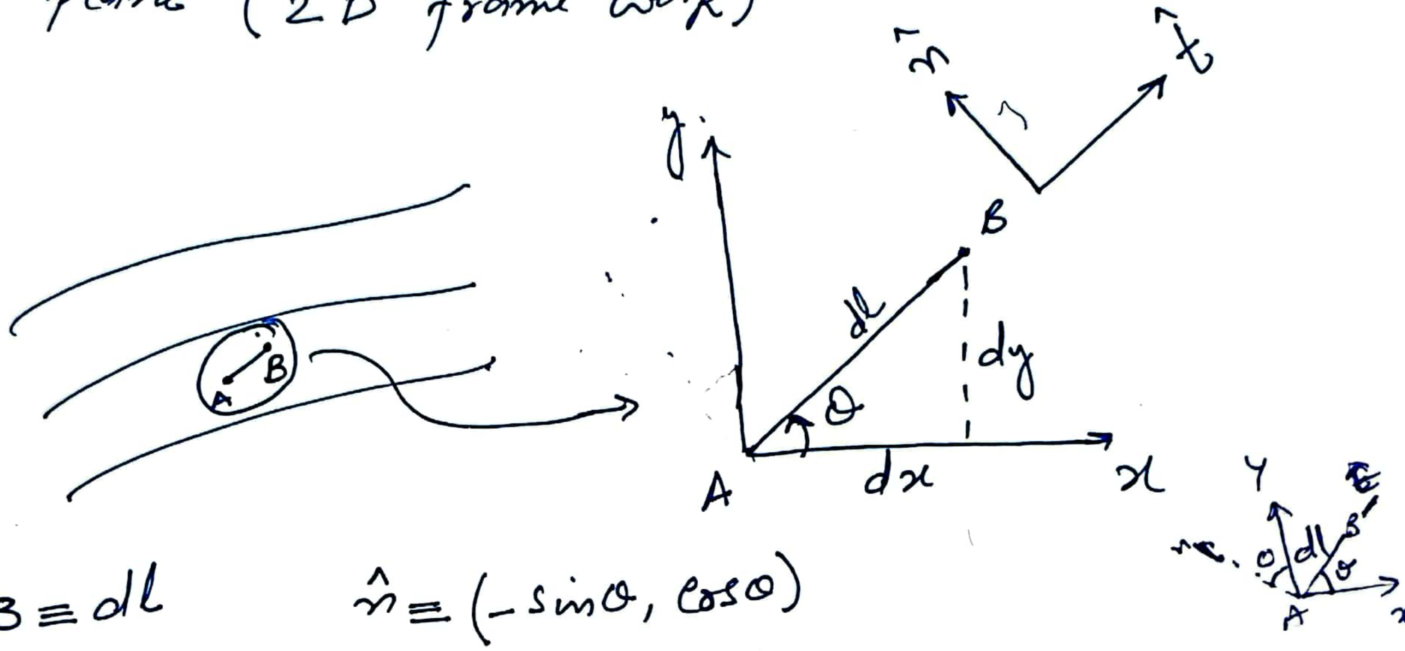
A time line or a material curve is a curve drawn in the flow domain at same time instant. The curve essentially represents instantaneous locations of a set of particles. In line as the constituent particles moves, a time line / material curve translates & deforms




Material curve at different time instants

## Deformation and Rotation rates

In order to visualized deformations, rotation rates we consider the kinematics of an infinitesimal material line anchored at some point in the flow domain. For the purpose of simplicity we consider the kinematics in a plane (2D frame work)




$$AB \equiv dl \qquad \hat{n} \equiv (-\sin\theta, \cos\theta)$$

$$\hat{t} \equiv (\cos\theta, \sin\theta)$$

$$\vec{V}_A \equiv (u_A, v_A)$$

$$\vec{V}_B \equiv \left[u_A + \left(\frac{\partial u}{\partial x}\right)_A dx + \left(\frac{\partial u}{\partial y}\right)_A dy \,,\; v_A + \left(\frac{\partial v}{\partial x}\right)_A dx + \left(\frac{\partial v}{\partial y}\right)_A dy\right]$$

$\left(\vec{V}_B - \vec{V}_A\right)_{\parallel AB}$ : responsible for deformation of AB.

$\left(\vec{V}_B - \vec{V}_A\right)_{\perp AB}$ : responsible to rotation of AB.

Instantaneous rotation rate of AB

$$\dot{\omega}_\theta = \frac{(\vec{V}_B - \vec{V}_A)\cdot\hat{n}}{dl}$$

$$= \frac{\left[\left(\frac{\partial v}{\partial x}\right)_A dx + \left(\frac{\partial v}{\partial y}\right)_A dy\right]\cos\theta - \left[\left(\frac{\partial u}{\partial x}\right)_A dx + \left(\frac{\partial u}{\partial y}\right)_A dy\right]\sin\theta}{dl}$$

$dx = dl\cos\theta$
$dy = dl\sin\theta$

$$\dot{\omega}_\theta = \left(\frac{\partial v}{\partial x}\right)_A \cos^2\theta + \left[\left(\frac{\partial v}{\partial y}\right)_A - \left(\frac{\partial u}{\partial x}\right)_A\right]\sin\theta\cos\theta - \left(\frac{\partial u}{\partial y}\right)_A \sin^2\theta \quad (1)$$

Deformation:

$$\dot{\epsilon}_\theta = \frac{(\vec{V}_B - \vec{V}_A)\cdot\hat{t}}{dl}$$

$$\dot{\epsilon}_\theta = \left(\frac{\partial u}{\partial x}\right)_A \cos^2\theta + \left[\left(\frac{\partial u}{\partial y}\right)_A + \left(\frac{\partial v}{\partial x}\right)_A\right]\sin\theta\cos\theta + \left(\frac{\partial v}{\partial y}\right)_A \sin^2\theta \quad (2)$$

at $\theta = 0$ & $\theta = 90$

$$\frac{1}{2}\left(\dot{\omega}_\theta + \dot{\omega}_{\theta+\frac{\pi}{2}}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_A = \frac{1}{2}\Omega \leftarrow \text{fluid vorticity.}$$

$$\dot{\epsilon}_{\theta=0} = \left(\frac{\partial u}{\partial x}\right)_A = \dot{\epsilon}_{xx}$$

$$\dot{\epsilon}_{\theta=\frac{\pi}{2}} = \left(\frac{\partial v}{\partial y}\right)_A = \epsilon_{yy}$$

$2\sin\theta\cos\theta = \sin 2\theta$
$\sin^2\theta = \frac{\sin 2\theta}{2}$
$\cos^2\theta = \frac{\cos 2\theta}{2}$

$$\dot{\gamma}_{\theta,\theta+\frac{\pi}{2}} = \dot{\omega}_\theta - \dot{\omega}_{\theta+\frac{\pi}{2}}$$

$$= \left(\frac{\partial v}{\partial x}\right)_A \cos 2\theta + \left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right]_A \sin 2\theta + \left(\frac{\partial u}{\partial y}\right)_A \cos 2\theta$$

$$= \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]_A \cos 2\theta + \left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right]_A \sin 2\theta$$

$$\dot{\gamma}_{0,\frac{\pi}{2}} = \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] = \dot{\gamma}_{xy}$$

$\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\gamma}_{xy}$ at point A describe the deformation properties at point A.

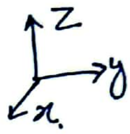
$$\epsilon_{i,j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \longrightarrow \text{strain rate tensor.}$$

$\gamma_{\theta, \theta+\frac{\pi}{2}}$

In 3D flows, the deformation and rotation rate of an infinitesimal material line.

<u>Deformation</u>: $\dot{\epsilon}_{xx} = \frac{\partial u}{\partial x}$, $\dot{\epsilon}_{yy} = \frac{\partial v}{\partial y}$, $\dot{\epsilon}_{zz} = \frac{\partial w}{\partial z}$

$$\dot{\gamma}_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \dot{\gamma}_{yz} = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \dot{\gamma}_{xz} = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$




<u>Rotation</u>: $\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$

Average rotation rate of any <u>two</u> $\perp$ material line in y-z plane

$$\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The six strain rate cartesian <u>components</u> are <u>represented</u> conventionally as a second rank symmetric tensor.

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \begin{matrix} i = 1, \dots, 3 \\ j = 1, \dots, 3 \end{matrix}$$

The vorticity components are combined into a vorticity vector $\vec{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$

$$\boxed{\vec{\Omega} = \nabla \times \vec{V}}$$

The strain rate tensor can also be written in a

# Unit 2. Fluid Kinematics.

## 2.1. Description of fluid Motion.

Joseph Louis, Compte de Lagrange (1736 - 1813)

### 2.1.1. Lagrangian description of motion.
(particle based)

In the lagrangian description of fluid motion each fluid particle is followed. i.e we focus our observation on a particle and follow it with time and study the changes in its properties ~~(mass~~ (velo., accn., temp. etc).

e.g. when ~~of~~ Lagrangian description is used, are ball falling from a height, pollution studies etc. Here position, velocity, accn. are specified as fn. of time only.

### 2.1.2. Eulerian description of motion.
Leonhard Euler (1707-1783)

(field based)

Here the properties of a flow field are described as functions of space coordinates and time. A fixed location is identified in the flow field and ~~flow~~ properties (eg. velo, accn. temp etc are studied as material passes through it.

Eulerian & Lagrangian description can be understood by the example of traffic study.

1. Using a fixed car (mounting it with instruments). [Lagrangian]
2. Identifying a fixed patch of road. [Eulerian].

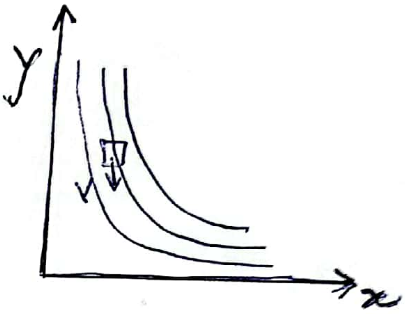
## 2.2 Fluid Acceleration in a Velocity field.

Translation of fluid particle is connected with the velocity field $\vec{V} = V(x,y,z,t)$. We need accn. for use in $\vec{F} = m\vec{a}$.

But $\vec{a} \neq \frac{\partial \vec{V}}{\partial t}$ since $\vec{V}$ is a field i.e it describes the whole flow and not just the motion of individual particle.

e.g flow through a corner.

Here particles are accelerating & decelerating i.e $\vec{a} \neq 0$ but $\frac{\partial \vec{V}}{\partial t} = 0$




So, given $\vec{V} = \vec{V}(x,y,z,t)$ find the accn. of a fluid particle $\vec{a_p}$

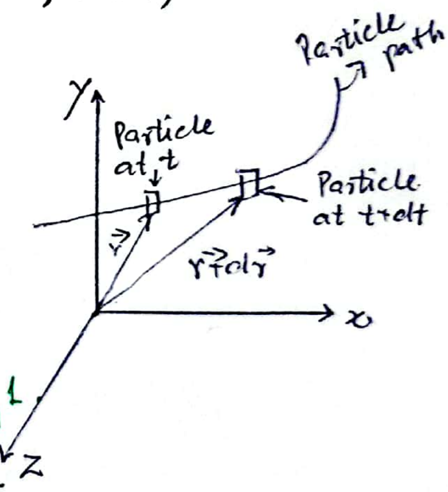
At $t$, $\quad \vec{V_p}\big|_t = \vec{V}(x,y,z,t)$

At $t+dt$

$$\vec{V_p}\big|_{t+dt} = \vec{V}(x+dx, y+dy, z+dz, t+dt)$$

The change in $\vec{V_p}$ in moving from $\vec{r}$ to $\vec{r}+d\vec{r}$

$$d\vec{V_p} = \frac{\partial \vec{V}}{\partial x}dx_p + \frac{\partial \vec{V}}{\partial y}dy_p + \frac{\partial \vec{V}}{\partial z}dz_p + \frac{\partial \vec{V}}{\partial t}dt$$




The total accn. of the particle is given by.

$$\vec{a_p} = \frac{d\vec{V_p}}{dt} = \frac{\partial \vec{V}}{\partial x}\underbrace{\frac{dx_p}{dt}}_{u} + \frac{\partial \vec{V}}{\partial y}\underbrace{\frac{dy_p}{dt}}_{v} + \frac{\partial \vec{V}}{\partial z}\underbrace{\frac{dz_p}{dt}}_{w} + \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt}$$

$$\vec{a_p} = \boxed{\frac{d\vec{V_p}}{dt}} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

$$\frac{D\vec{V}}{Dt} \; ; \; \frac{D}{Dt} = \left(\underbrace{\boxed{u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}}}_{(\vec{V}\cdot\vec{\nabla})} + \frac{\partial}{\partial t}\right)$$

Material or Substantial derivative.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla})$$

# $\frac{\partial}{\partial t}$ is the [temporal or local derivative] which expresses rate of change with time at a fixed position.

# $\vec{v} \cdot \vec{\nabla}$ is known as [convective derivative] which represents the time rate of change due to change in position in the field.

Remarks:-

1. In a steady flow, $\frac{\partial}{\partial t} = 0$.
2. In a uniform flow, $(\vec{v} \cdot \vec{\nabla}) = 0$.
3. In a steady uniform flow, $\frac{D}{Dt} = 0$

## 2.3 Steady & Unsteady Flow

In steady flow the hydrodynamic parameters may vary with location. but the spatial distribution of these parameters remain invariant with time.

Note! For steady flow, the description of flow under both the eulerian & the Lagrangian approaches become identical.

## 2.4. Uniform & non Uniform flow

For uniform flow the hydrodynamic parameters do not change from point to point at any instant of time.

1. Steady-Uniform → Flow at a constt. rate through a duct of uniform cross-section. (wall effects disregarded)

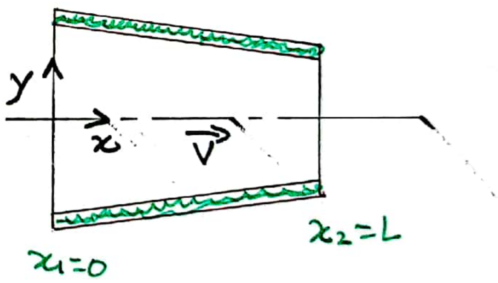
2. Steady-non-uniform flow → Flow at constt. rate through a duct of non-uniform X-section. [tapering pipe]

3. Unsteady uniform flow → Flow at varying rates through a pipe of uniform X-section. [no wall effects)

4. Unsteady-non-uniform flow → Flow at varying rates through a tapered pipe.

**Prob.** Particle description in Eulerian & Lagrangian approach.

Consider a 2D, steady, incompressible flow through the plane converging channel shown. The velo. along x-axis is given by $\vec{V} = V_1[1+(x/L)]\hat{i}$. Find the accn. for the particle moving along that centerline. If we use the method of description of particle mechanics, the




position of the particle, located at $x=0$ at $t=0$, will be a function of time, $x_p = f(t)$. Obtain the exp. of $f(t)$ and then by taking $\frac{d^2}{dt^2}$ obtain x-comp. of particle accn.

Given: Steady, 2D, incomp. flow

$$\vec{V} = V_1\left[1 + \frac{x}{L}\right]\hat{i}$$

Find; a) particle accn. along $x$.

b) For a particle at $x=0, t=0$, expression for.

1) position $x_p$, as a fn of time

2) $x$ comp. of accn. $a_{xp}$.

Solu:-

a) The accn. of a particle moving in the velo. field is given by.

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

The $x$- comp of accn. of particle is given by.

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

For any particle on $x$-axis. $v=w=0$; $u = V_1\left[1+\frac{x}{L}\right]$

Therefore, $\frac{Du}{Dt} = u\frac{\partial u}{\partial x} = V_1\left(1+\frac{x}{L}\right)\frac{V_1}{L} = \frac{V_1^2}{L}\left(1+\frac{x}{L}\right)$ } Eulerian description.

By substituting position of particle in above eq. we can obtain its accn.

b) Now we want to follow a single particle which is located at $x=0$ at $t=0$.

The $x$- coor. that locates this particle will be a fn. of time. $x_p = f(t)$. Also, $u_p = df/dt$. again a fn. of time. At $t=0$, $x=0$. $\therefore$ $u_p = V_1$; At some time, $t$, $x=L$, so $u_p = 2V_1$

To find $x_p = f(t)$. we write

$$V = \frac{dx_p}{dt} = \frac{df}{dt} = V_1\left[1+\frac{x}{L}\right] = V_1\left[1+\frac{f}{L}\right]$$

or $$\frac{df}{(1+f/L)} = V_1 dt$$

At $t=0,\ x=0,$ at $t=t,\ x=f.$

$\therefore$ Integrating above eq. and applying limits.

$$\int_0^f \frac{df}{(1+f/L)} = \int_0^t V_1 dt \quad \text{or} \quad L\ln\left(1+\frac{f}{L}\right) = V_1 t$$

Then, $$\ln\left[1+\frac{f}{L}\right] = \frac{V_1 t}{L} \quad \text{or} \quad 1+\frac{f}{L} = e^{V_1 t/L}$$

and $$f = L\left[e^{V_1 t/L} - 1\right].$$

The position of particle, located at $x=0$ at $t=0$, is given by

$$x_p = f(t) = L\left[e^{V_1 t/L} - 1\right].$$

x- comp. of accn.

$$a_{x_p} = \frac{d^2 x_p}{dt^2} = \frac{d^2 f}{dt^2} = \frac{V_1^2}{L} e^{V_1 t/L}$$ } Lagrangian description

Lag:

a) At $x=0, t=0$

$$a_{x_p} = \frac{V_1^2}{L}$$

b) At $x_p = L/2$, $t=t_1$

$$x_p = L/2 = L\left[e^{V_1 t_1/L} - 1\right] \text{ or } e^{V_1 t_1/L} = 1.5$$

$$a_{x_p} = \frac{V_1^2}{L} e^{V_1 t_1/L} = 1.5\frac{V_1^2}{L}$$

Eul

a) $$a_{x_p} = \frac{Du}{Dt} = \frac{V_1^2}{L}(1+0) = \frac{V_1^2}{L}$$

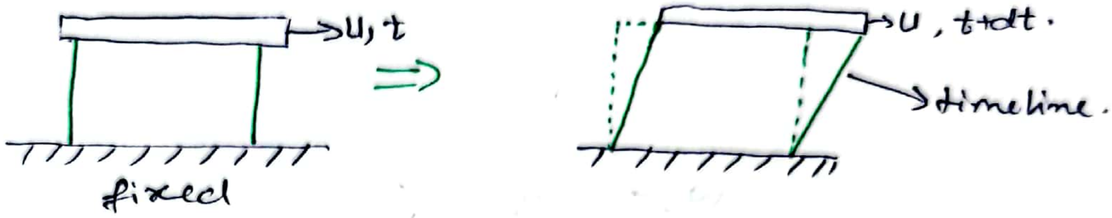
b) At $x = 0.5L$

$$\frac{Du}{Dt} = \frac{V_1^2}{L}(1+0.5) = \frac{1.5V_1^2}{L}$$

# Flow Visualisation.

**Timeline:-** If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line in the fluid at that instant; called as timeline.

eg.




**Path line:** It is the path traced out by a moving fluid particle.

**Streak line:** If we focus our attention on a fixed location in space and identify by using dye or smoke all the fluid particles passing through this point. The line joining these fluid particles is defined as a streakline.

**Stream lines:** are lines drawn in the flowfield so that at a given instant they are tangent to the direction of flow at every point in the flow field. Since the streamlines are tangent to $\vec{V}$. there can be no flow across a stream line.

In a steady flow, path lines, stream lines and streak lines are identical lines in the flow field. In case of unsteady flow they do not coincide.

# Calculation of Streamlines, pathlines & Streaklines

**Streamline:-** By definition $\vec{V} \times d\vec{r} = 0$ which yields the eq. of streamlines for a given time instant. $t = t_1$

Let $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, $dr = dx\hat{i} + dy\hat{j} + dz\hat{k}$.

So, $\dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w} = ds$, $s$ = integration parameter

So if $(u, v, w)$ are known, integrate w.r.t $s$ for $t = t_1$ with initial condition $(x_0, y_0, z_0, t_0)$ at $s = 0$, then eliminate $s$.

**Path Line:-** Path line is defined by integration of the relationship between velocity and displacement.

$$\frac{dx}{dt} = u \; ; \quad \frac{dy}{dt} = v \quad \frac{dz}{dt} = w.$$

Integrate $u, v, w$ w.r.t $t$ using I.C $(x_0, y_0, z_0, t_0)$ then eliminate $t$.

**Streakline!** Use the path line eq. retaining time as a parameter. Now, find the integration constt. which causes the path line to pass through $(x_0, y_0, z_0)$ for a sequence of time $\tau < t$. Then eliminate $\tau$.

Problem:- Given ~~a~~ the velo distribution $u = \frac{x}{1+t}$, $v = \frac{y}{1+2t}$, $w = 0$.

Calculate 1) Streamlines 2) pathlines 3) Streaklines which pass through $(x_0, y_0, z_0)$ at $t=0$.

Solu:- 1) Since $w=0$, $\therefore$ 2-D flow.

$$\frac{dx}{ds} = u = \frac{x}{1+t} \qquad \frac{dy}{ds} = v = \frac{y}{1+2t}$$

Integrate

$$x = C_1 \exp\left(\frac{s}{1+t}\right) \qquad y = C_2 \exp\left(\frac{s}{1+2t}\right)$$

$s=0$ at $(x_0, y_0)$; $\quad C_1 = x_0, \quad C_2 = y_0$.

eliminating $s$;

$$s = (1+t)\ln\frac{x}{x_0} = (1+2t)\ln\frac{y}{y_0}$$

$$\therefore \quad y = y_0\left(\frac{x}{x_0}\right)^{\left(\frac{1+t}{1+2t}\right)}$$

This is the eq. of streamline which passes through $(x_0, y_0)$ for all times $t$.

$$t=0, \quad \frac{y}{y_0} = \frac{x}{x_0}$$

$$t=\infty, \quad \frac{y}{y_0} = \left(\frac{x}{x_0}\right)^{1/2}$$

2) To find path lines we integrate.

$$\frac{dx}{dt} = u = \frac{x}{1+t} \qquad \frac{dy}{dt} = v = \frac{y}{1+2t}$$

Integrate

$$\int\frac{dx}{x} = \frac{dt}{1+t} \qquad \frac{dy}{y} = \frac{dt}{1+2t}$$

or $x = C_1(1+t)$ $\qquad y = C_2(1+2t)^{1/2}$

using. $\int\frac{c}{ax+b}dx = \frac{c}{a}\ln(ax+b) + C$

At $t=0$, $(x,y) = (x_0, y_0)$ we get

$C_1 = x_0 \qquad C_2 = y_0$

Now eliminate $t$ b/w the equations for $(x, y)$

$$y = y_0 \left[1 + 2\left(\frac{x}{x_0} - 1\right)\right]^{1/2}$$

This is the pathline through $(x_0, y_0)$ at $t=0$ and does not coincide with the streamline at $t=0$.

3 To find streakline, we use the pathline eq. to find the family of particles that have passed through the pt. $(x_0, y_0)$ for all times $\tau < t$.

$$x = c_1(1+t) \qquad y = c_2(1+2t)^{1/2}$$

$$x_0 = c_1(1+\tau) \qquad y_0 = c_2(1+2\tau)^{1/2}$$

$$c_1 = \frac{x_0}{1+\tau} \qquad c_2 = \frac{y_0}{(1+2\tau)^{1/2}}$$

$$\therefore \; x = \frac{x_0}{1+\tau}(1+t) \qquad y = \frac{y_0}{(1+2\tau)^{1/2}}(1+2t)^{1/2}$$

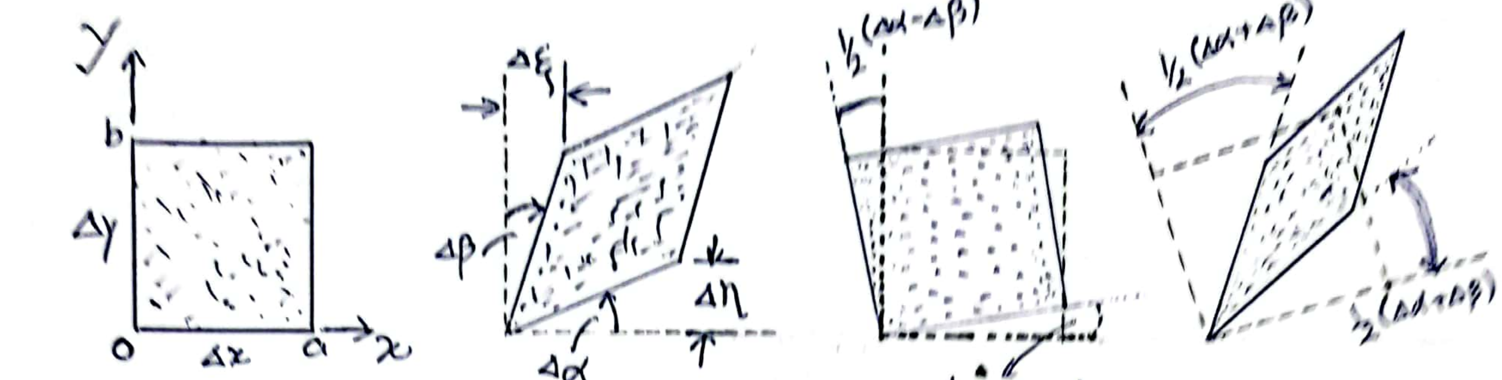
$$\therefore \; \tau = (1+t)\frac{x_0}{x} - 1; \qquad \Rightarrow \frac{1}{2}\left[(1+2t)\left(\frac{y_0}{y}\right)^2 - 1\right]$$

$$\text{or} \; \left(\frac{y_0}{y}\right)^2 = \frac{1+2t}{1+2\left[(1+t)\left(\frac{x_0}{x} - 1\right)\right]}$$

for $t=0$; $\frac{y}{y_0} = \left[1 + 2\left(\frac{x_0}{x} - 1\right)\right]^{-1/2}$

The streakline does not coincide with either the equivalent pathline or streamline

# Fluid rotation.




(a) Original particle (b) Particle after time Δt (c) Rotational component (d) Angular deformation component.

Consider the xy plane view of the particle at time t. Shown in (a). After Δt the particle in general would have translated, rotated and deformed. Line oa rotated CCW by Δα and line ob rotated CW by Δβ. [Using right hand rule. consider CCW as +ve rotation]

The particle's rigid body CCW rotation is taken as the average of the rotations Δα & Δβ. $\doteq \frac{1}{2}[\Delta\alpha + (-\Delta\beta)]$ ↳ due to CW dir.

The particle's deformation can be obtained by fig (b) & (c) If we subtract the particle rotation $\frac{1}{2}(\Delta\alpha - \Delta\beta)$ from the actual rotation of oa, we will get pure deformation.

$$= \Delta\alpha - \frac{1}{2}(\Delta\alpha - \Delta\beta) = \frac{1}{2}(\Delta\alpha + \Delta\beta).$$

Angular velocity of the particle about z axis; $\omega_z$.

$$\omega_z = \lim_{\Delta t \to 0} \frac{\frac{1}{2}(\Delta\alpha - \Delta\beta)}{\Delta t}$$ [$\omega_z$ = rate of rotation]

From fig (a) & (b) for small angles.

$\Delta\alpha = \Delta h / \Delta x$ $\qquad\qquad$ $\Delta\beta = \Delta l_q / \Delta y$

In time $\Delta t$ point 'o' moved horizontally $u\Delta t$, the point 'b' will have moved ~~as~~ $[u + (\partial u/\partial y)\Delta y]\Delta t$. Similarly, vertical movement of 'o' is $v\Delta t$, then point 'a' will have moved ~~as~~ $[v + (\partial v/\partial x)\Delta x]\Delta t$.

$$\therefore \Delta l_q = \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta t - u\Delta t = \frac{\partial u}{\partial y}\Delta y \,\Delta t$$

$$\Delta h = \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t = \frac{\partial v}{\partial x}\Delta x \,\Delta t$$

Now angular velocity $\omega_z$ will be

$$\omega_z = \underset{\Delta t \to 0}{Lt} \frac{\frac{1}{2}(\Delta\alpha - \Delta\beta)}{\Delta t} = \underset{\Delta t \to 0}{Lt} \frac{\frac{1}{2}\left(\frac{\Delta h}{\Delta x} - \frac{\Delta l_q}{\Delta y}\right)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{Lt} \frac{\frac{1}{2}\left(\frac{\partial v}{\partial x}\frac{\Delta x}{\Delta x}\Delta t - \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta y}\Delta t\right)}{\Delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Similarly,

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \text{and} \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Then, $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$ becomes.

$$\vec{\omega} = \frac{1}{2}\underbrace{\left[\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]}_{\text{Curl } \vec{V} = \nabla \times \vec{V}}$$

∴ In vector notation, particle rotation is.

$$\vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{V}$$

Note:- Particle rotation should not be confused with flow consisting of circular streamlines (rigid body rotation), or vortex flow.

1. Rotation of fluid particles will always occur for flows in which we have shear stresses. Shear stresses are present in viscous flows.

2. Flows for which no particle rotation occurs are called Irrotational flows. i-e, $\nabla \times \vec{V} = 0$

3. No real fluid is truly irrotational [presence of $\mu$], but many flows can be successfully studied by assuming they are inviscid & irrotational.

Vorticity. ($\vec{\Omega}$)

Vorticity is a measure of the rotation of a fluid element as it moves in a flow field.

$$(\vec{\Omega}) = 2\vec{\omega} = \nabla \times \vec{V}$$

Biren

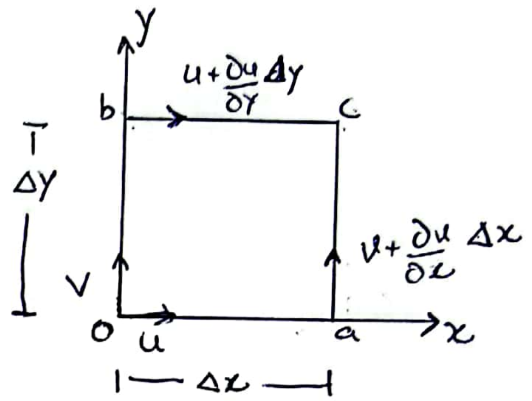
## Circulation.

The circulation ($\Gamma$) is defined as the line integral of the tangential velocity component about any closed curve fixed in the flow.

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s}$$

[ +ve sense corresponds to CCW.

For the closed curve oacb,




$$\Delta\Gamma = u\Delta x + \left(v+\frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u+\frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\Delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\Delta y$$

or $\Delta\Gamma = 2\omega_z \Delta x \Delta y$

Then,

$$\Gamma = \oint_c \vec{V}\cdot d\vec{s} = \int_A 2\omega_z\, dA = \int_A (\nabla\times\vec{V})_z\, dA.$$

Above eq. is a statement of the Stokes Theorem in 2D. Thus, circulation around a closed contour is equal to the total vorticity enclosed within it.

## Angular Deformation.

As discussed before, the angular deformation is given by $(\Delta\alpha+\Delta\beta)$. $\therefore$ Rate of angular deformation in the xy plane is given by

$$\underset{\Delta t\to 0}{Lt}\ \frac{(\Delta\alpha+\Delta\beta)}{\Delta t} = \underset{\Delta t\to 0}{Lt}\ \frac{\left(\frac{\partial v}{\partial x}\frac{\Delta x}{\Delta x}\Delta t + \frac{\partial u}{\partial y}\frac{\Delta y}{\Delta y}\Delta t\right)}{\Delta t}$$

contd - - - -

Rate of angular deformation in xy plane $= \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$ ——— (A)

Similarly, rate of angular deformation in yz & zx plane is given by; $\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$ and $\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$ respectively.

Note: In Unit 1, while discussing 1-D, laminar Newtonian flow, we found shear stress ($\tau$) $\propto$ rate of deformation $\left(\frac{\partial u}{\partial y}\right)$ of the fluid particle.

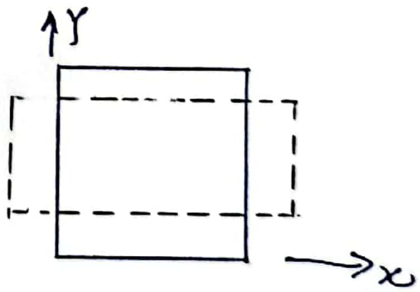
$$\tau = \mu\left(\frac{\partial u}{\partial y}\right) \quad \text{——— (B)}$$

Rate of deformation used in eq. (B) is a special case of eq. (A)

## Linear Deformation.

During linear deformation, the angle at the vertices of the fluid element, remains unchanged.

The element will change in x, y and z only if $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z} \neq 0$.




where $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$ represent longitudinal rate of strain in the x, y, z dir. resp.

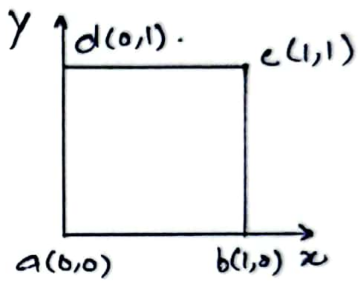
The rate of local instantaneous volume dilatation is given by.

$$\text{Volu. strain rate} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla . \vec{V}$$

For incompressible flow, the rate of volume dilatation is zero. i.e $\nabla \cdot \vec{V} = 0$

**Note:** If we know the velocity field, we can determine the accn., rotation, angular deformation, and linear deformation of a fluid particle in a flow field.

**Problem:** Two-Dim. flow field $\vec{V} = Axy\hat{i} + By^2\hat{j}$, where $A = 1\,m^{-1}s^{-1}$; $B = -\frac{1}{2}\,m^{-1}s^{-1}$. Show flow is incompressible and find rotation at $(x,y) = (1,1)$. Find circulation about the curve shown.




Solu. For. incompr. flow $\nabla \cdot \vec{V} = 0$

or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial}{\partial x}(Axy) + \frac{\partial}{\partial y}(By^2) = Ay + 2By = (1)y + 2(-\tfrac{1}{2})y = 0.$$

Fluid rotation is given by. $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ Axy & By^2 & 0 \end{vmatrix} = -\frac{1}{2}Ax\hat{K}$$

$$\vec{\omega}_{1,1} = -\frac{1}{2}(1)(1)\hat{k} = -0.5\hat{k} \text{ rad/s}$$

$$\text{Circulation }(\Gamma) = \oint \vec{V} \cdot d\vec{s} \qquad = \int_d^c u(-dx)$$

$$\Gamma = \int_a^b \underbrace{u\,dx}_{u=0 \text{ along } ab} + \int_b^c v\,dy + \int_c^d u(dx) + \int_d^a v(dy)$$

$$= \int_0^1 By^2\,dy + \int_1^0 \underbrace{Axy\,dx}_{y=1 \text{ along } cd} + \int_1^0 By^2\,dy = -\frac{1}{2}A = -\frac{1}{2}\,m^2/s.$$

# Streamfunction

The use of streamfunction can considerably simplify the description of incompressible 2D flow. It is a function which satisfies the law of mass conservation.

Volumetric strain is given by.

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{\partial u_i}{\partial x_i} \qquad i = x, y, z. \quad —(1)$$

$\frac{D}{Dt}$ signifies that a specific particle is followed.

$$\therefore \quad \delta V \propto \frac{1}{\rho} \quad —(2)$$

From (1) & (2)

$$-\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{\partial u_i}{\partial x_i} \quad —(3)$$

Eq(3) is called continuity eq.

Under certain conditions of fluid flow the density variations in the flow are small. The most important condition is that; $M \ll 1$, or $V \ll a$,. $M < 0.3$.(approx.). This is called the Boussinesq approximation. Due to this condition.

$$\frac{1}{\rho}\frac{D\rho}{Dt} \ll \frac{\partial u_i}{\partial x_i} \quad —(4)$$

or $$\frac{\partial u_i}{\partial x_i} = 0 \quad —(5)$$

Eq(5) gives continuity eq for both steady and unsteady condition. Generally for 2D flows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad —(6)$$

Now if we define $\psi(x,y,t)$ such that

$$u \equiv \frac{\partial \psi}{\partial y} \quad ; \quad v \equiv -\frac{\partial \psi}{\partial x} \qquad \text{— (7)}$$

Eq (3) is satisfied.

Note: For compressible flows a streamfunction can be defined if the motion is 2D and steady.

Exercise: Consider a steady axisymmetric flow of a compressible fluid in $(R, x, \theta)$.

$$\frac{\partial}{\partial R}(\rho R u_R) + \frac{\partial}{\partial x}(\rho R u_x) = 0$$

Show how we can define a streamfunction so that the eq. of continuity is satisfied automatically.

- One purpose: Streamfunction enable us to plot streamlines.
- Second: It decreases the number of simultaneous equations which are to be solved for defining a flow. For eg. the momentum & mass conservation eq for viscous flows near a planar solid boundary are given by.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad - \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Using $\psi$, the momentum eq. can be written as.

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}.$$

Now instead of two, one eq. can describe the flow, as $\psi$ automatically satisfies continuity.

# Stream function.

⇒ Stream function allows us to mathematically represent two entities; the velo. comp. $u(x,y,t)$ & $v(x,y,t)$ of a 2D incompressible flow, using a single function $\psi(x,y,t)$

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial x}$$

Above definition of $\psi$ guarantees that any continous fn $\psi(x,y,t)$ automatically satisfies the 2D. incompressible continuity eq.

i.e. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ [We have seen this eq. before]

So, $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

We know that at each point the stream lines are tangent to the instantaneous velo. vecter. i.e

$$\frac{dy}{dx} = \frac{v}{u}.$$

Thus eq. of streamline in 2D is

$$u\,dy - v\,dx = 0$$

or $$\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = 0 \quad \text{——— (A)}$$

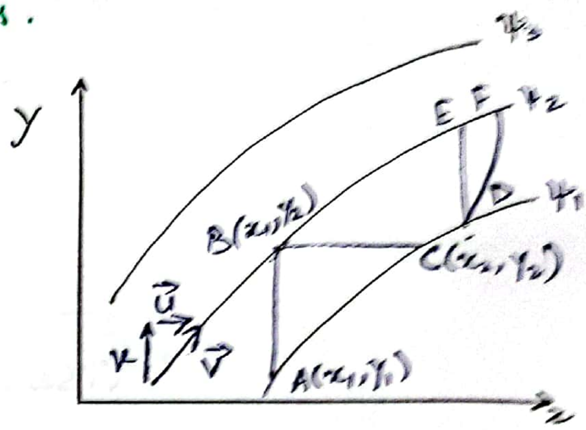
Also, change in $\psi$ is given by

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \quad \text{——— (B)}$$

Comparing (A) & (B), we can say that along an instantaneous Streamline $d\psi = 0$. i.e $\psi$ = constt along a streamline.

Thus, each value of $\psi$ represents a stream line.

**Significance of $\psi$:** They can be used to obtain the volume flow rate between two stream lines.




Vol. flow rate can be calculated b/w streamlines $\psi_1$ & $\psi_2$, by using line AB, BC, DE or ~~BF~~ DF

Flow rate using line AB. [considering unit depth $\perp$ xy plane].

$$Q = \int_{y_1}^{y_2} u\,dy \cdot 1 = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\,dy$$

$x$ is constt. along AB. $\therefore\ d\psi = \frac{\partial \psi}{\partial y}\,dy$.

$$\therefore\ Q = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y}\,dy = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

Similarly, for unit depth, the flow rate across BC is

$$Q = \int_{x_1}^{x_2} v\,dx = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x}\,dx$$

$y$ is constt. along BC, $d\psi = \partial\psi/\partial x\,dx$.

$$\therefore\ Q = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x}\,dx = -\int_{\psi_2}^{\psi_1} d\psi = \psi_2 - \psi_1$$

The volume flow rate (per unit depth) between two streamlines can be written as the difference b/w the constt. values of $\psi$ defining the two streamlines.

Note:· Since the volume flow b/w two streamlines is constt.
the velo. will be relatively high whenever the streamlines are close together, and relatively low whenever they are far apart.

For 2D. incompressible flow, in r.θ. plane. Cons. of mass (volumetric strain) is given by.

$$\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_\theta}{\partial \theta} = 0$$

So. the stream function $\psi(r,\theta,t)$, is given by

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r}$$

Also, for 2D. steady compressible flow. in xy plane. $\psi$ can be defined such that

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho v = -\frac{\partial \psi}{\partial x}$$

Now, the difference b/w the constt values of $\psi$ defining two streamlines is the mass flow rate (per unit depth). b/w the two streamlines.

Prob: Velo. field for steady, incompressible flow in a corner is given by $\vec{V} = Ax\hat{i} - Ay\hat{j}$ ; $A = 0.3\,s^{-1}$. Determine the stream fn. for this velo. field. Plot the streamlines in I$^{st}$ and II$^{nd}$ quadrant in xy plane. Find Q? b/w any two streamlines

Soln. The flow is incompressible.

$$\therefore\ u = \frac{\partial \psi}{\partial y} = Ax, \qquad v = -\frac{\partial \psi}{\partial x} = -Ay$$

Integrating w.r.t $y$.

$$\psi = \int \frac{\partial \psi}{\partial y}\, dy + f(x) = \boxed{Axy + f(x)} \quad [f(x) \text{ is arbitrary}] \qquad \text{——— (1)}$$

Where, $f(x)$ may be evaluated using eq. for $v$.

$$v = -\frac{\partial \psi}{\partial x} = -Ay - \frac{df}{dx} \qquad \text{——— (2)}$$

From the velo field. $v = -Ay$. Comparing this with eq (2) gives $\frac{df}{dx} = 0$. or $f(x) = $ constt.

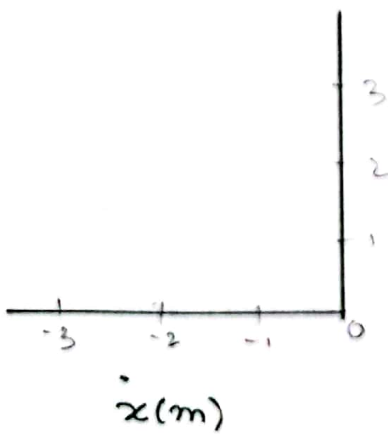
$$\therefore\ \psi = Axy + C$$

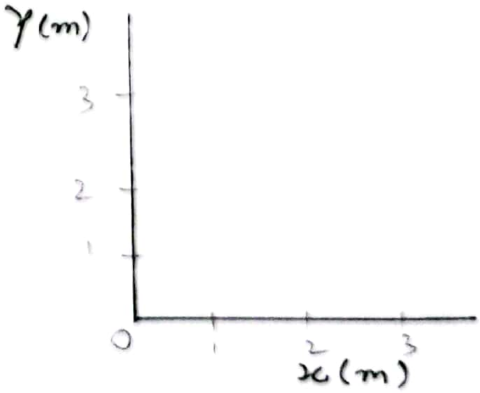
Since each constt value of $\psi$ represents a streamline. we can choose different values of 'C' to obtain different $\psi$. For $C=0$ the streamline through origin, $\psi = \psi_1 = 0$.

$$\underbrace{\psi = 0.3}_{=A}\, xy \quad (m^3/s/m)$$

[complete the plot below, & comment on u, v - dir. in Q1 & Q2, also, find vol. flow rate b/w any two stream lines [to be done by class]

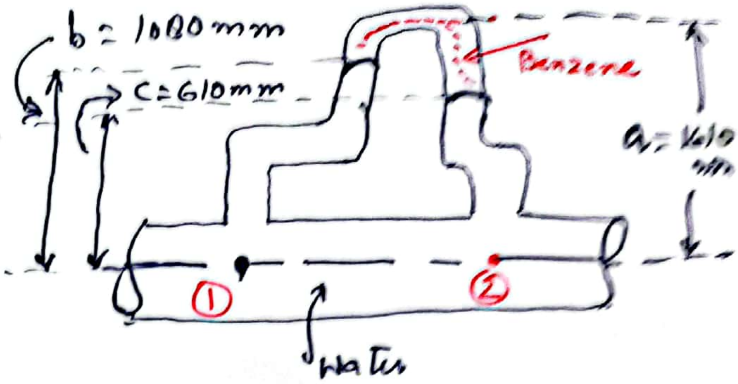
Q2. Q1



# Manometer:

① Find $P_1 - P_2$




## Streamline, Streakline, pathline.

\# $\vec{V} = axt\,\hat{i} + b\hat{j}$ , $a = 0.3\ s^{-1}$, $b = 2\ m/s$.

1. Find & plot the pathline during $0 \le t \le 3s$. of the particle that passed through the pt $(x_0, y_0) = (1, 2)$ at $t = 0$.

2. Find streakline through the same pt. at $t = 3s$.

Soln:

$$u = \frac{dx}{dt} = axt \qquad v = \frac{dy}{dt} = b$$

$$\int_{x_0}^{x} \frac{dx}{x} = \int_{t_0}^{t} t\,dt \Rightarrow \ln\frac{x}{x_0} = \frac{1}{2}a\left(t^2 - t_0^2\right)$$

$$\Rightarrow x = \cancel{x_0 e^{\frac{1}{2}at^2}}\ x_0 e^{\frac{1}{2}at^2}$$

b) $$x = x_0 e^{\frac{1}{2}a\left(t^2 - t_0^2\right)}$$